Optical soliton propagation in a coupled system of the nonlinear Schrödinger equation and the Maxwell-Bloch equations

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Abstract. Optical soliton propagation in fibres with resonant impurities and erbium-doped nonlinear systems are considered. The fibre system is governed by the coupled system of the nonlinear Schrödinger equation and the Maxwell-Bloch equations. The system equations using Painlevé analysis are found to be completely integrable, but only when the parameters involving the group velocity dispersion, Kerr nonlinearity and the field interaction with the atoms, satisfy a particular condition. With a suitable approximation, for the first time Painlevé analysis of the system equation is reported in this paper. The Lax pair of the system equation is explicitly constructed. The single soliton solution is also explicitly derived using the Darboux-Bäcklund transformation method.

1. Introduction

At present, many theoretical works concentrate more on the practical feasibility of optical solitons. One such important practically realizable system is the coupled version of the nonlinear Schrödinger (NLS) equation and Maxwell-Bloch (MB) equations. Very recently many researchers have worked on this, achieving many results [1-6].

The coupled NLS-MB system was proposed for the first time by Maimistov and Manykin [2] to treat ultra-short-pulse propagation in a light pipe with a two-level resonant medium with Kerr nonlinearity.

In optical fibres, two types of soliton are possible. One governed by the NLS equation which is a balance between the group velocity dispersion (in the anomalous dispersion regime) and the self-phase modulation due to the Kerr nonlinearity. The resulting dynamic equation for the NLS solitons is of the form [5-7]

\[ E_z = i \left( \frac{1}{2} E_{tt} + |E|^2 E \right), \]

where \( E(z, t) \) is the slowly varying electric field, and subscripts \( z \) and \( t \) denote partial derivatives. The NLS equation also finds application in the stationary two-dimensional self-focusing of a plane wave, the self-trapping phenomena of nonlinear optics and the propagation of a heat pulse in a solid [8].

The other possible soliton is due to the presence of two-level resonance medium (impurities or erbium) in the fibre core. This is governed by the MB equations given by [8]

\[ E_z = \langle p \rangle, \]
\[ p_t = i \alpha p - f E \eta, \]
\[ \eta_t = 2f (E p^* + E^* p), \]

(2)

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where \( p = v_1v_2^* \), \( \eta = |v_2|^2 - |v_1|^2 \) and \( f \) describe the character of interaction between the propagating field and the two level resonant atoms. Here \( v_1 \) and \( v_2 \) are the wavefunctions in a two-level system \( \omega \) is the frequency and \( \langle p \rangle \) stands for averaging with respect to inhomogeneous broadening of the resonant frequency;

\[
\langle p(z, t; \omega) \rangle = \int_{-\infty}^{\infty} p(z, t; \omega) g(\omega) \, d\omega
\]

\[
\int_{-\infty}^{\infty} g(\omega) \, d\omega = 1
\]

where \( g(\omega) \) is the uncertainty in the energy level (line-shape function). The pulse propagation described by the MB equations is often called the self-induced-transparency (SIT) solitons (for \( f = 1 \)).

In the practical case if one considers a soliton governed by the NLS equation, an impurity-free optical guide is essential. Otherwise, that will contribute to the radiation absorption spectrum [9]. Owing to the inhomogeneous broadening of the impurity energy levels there is always a group of levels that are at resonance with the radiation transmitted through the fibre. This will make the system contribute to the SIT. The impurity may also be erbium doped in the fibre for the amplification of the optical pulses, which tend to decay in course of its propagation. So, the exact version of the dynamics of the wave propagation in the fibre system is described by the NLS–MB coupled system.

In the derivation of the wave equation from the Maxwell equations, the polarization consists of two parts, namely the linear and the nonlinear polarizations. The linear polarization is due to the linear susceptibility \( \chi^{(1)} \) and contributes to the optical losses and the chromatic dispersion. To compensate for these losses, erbium-doped fibres are used which will produce the SIT in the nonlinear part. The nonlinear part of the polarization is caused by the next possible higher-order susceptibility \( \chi^{(3)} \) which will contribute to the Kerr nonlinearity and the SIT. For both the effects of \( \chi^{(3)} \) the detailed derivation of the NLS–MB system has been given in [3,4]. The system of equation is found to be

\[
E_z = ikE_{tt} - ig|E|^2 E + \langle p \rangle, \\
p_t = i\omega p - fE\eta, \\
\eta_t = 2f(E\eta^* + E^*p),
\]

where \( g \) is the Kerr nonlinearity parameter and \( k \) is the group velocity dispersion parameter.

The main aim of this paper is to investigate the soliton-type propagation of the NLS–MB system through the Painlevé analysis.

2. The Painlevé analysis

The Painlevé analysis is a powerful method in nonlinear science for establishing the integrability of a given nonlinear partial differential equation, that is solutions which are free from moveable critical manifolds [10,11]. In recent years, this method is mainly used to construct the integrability properties such as the Lax pair, the Bäcklund transformations, bilinear form, and soliton solutions [12,13].

Because of the averaging term \( \langle p \rangle \), as such, equation (4) cannot be studied from the Painlevé analysis point of view. Thus for mathematical convenience, the
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The line-shape function $g(\omega)$ in equation (3) is considered as a Dirac delta function at resonant frequency $\omega_0$. So, the averaging function reduces to

$$\langle p(z, t; \omega) \rangle = \int_{-\infty}^{\infty} p(z, t; \omega) \delta(\omega - \omega_0) \, d\omega$$

$$= p(z, t; \omega_0).$$

(3a)

This assumption on $g(\omega)$ is followed to avoid mathematical complexity. The complete Painlevé analysis of the exact system with the averaging function on $\rho(p)$ will be published elsewhere. Also the independent variables $z$ and $t$ are interchanged for mathematical simplicity. So, the new set of equations whose Painlevé analysis is going to be considered are

$$E_t = ikE_{zz} - ig|E|^2 E + p,$$

$$p_z = i\omega_0 p - fE\eta,$$

$$\eta_z = 2f(Ep^* + E^*p).$$

(5)

It is obvious that the variables $E$ and $p$ are complex and $\eta$ is real.

To apply the Painlevé analysis we set $E = a$, $E^* = b$, $p = c$, $p^* = d$ and $\eta = e$ and the resulting set of equations are

$$a_t = ika_{zz} - i\alpha a^2 b + c,$$

$$b_t = -ikb_{zz} + igb^2 a + d,$$

$$c_z = i\omega_0 c - fae,$$

$$d_z = -i\omega_0 d - fbe,$$

$$e_z = 2f(ad + bc).$$

(6)

The generalized Laurent expansions of $a, b, c, d$ and $e$ are

$$a = \phi^x \sum_{j=0} a_j(z, t) \phi^j,$$

$$b = \phi^\beta \sum_{j=0} b_j(z, t) \phi^j,$$

$$c = \phi^\gamma \sum_{j=0} c_j(z, t) \phi^j,$$

(7)

$$d = \phi^\delta \sum_{j=0} d_j(z, t) \phi^j,$$

$$e = \phi^\mu \sum_{j=0} e_j(z, t) \phi^j,$$

with $a_0, \ldots, c_0 \neq 0$; $\alpha, \beta, \gamma, \delta$ and $\mu$ are negative integers; $a_j, \ldots, e_j$ are the set of expansion coefficients which are analytical in the neighbourhood of the non-characteristic singular manifold $\phi(z, t) = z + \dot{\phi}(t) = 0$ [11]. Looking at the leading order, we substitute $a = a_0 \phi^x, \ldots, e = e_0 \phi^\mu$ in equation (6) and upon balancing different terms we obtain the following results:

$$\alpha = \beta = -1, \quad \gamma = \delta = \mu,$$

$$a_0 b_0 = -1, \quad a_0 d_0 = b_0 e_0.$$
Substituting the full expansion of the Laurent series in equation (6) and keeping the leading-order terms alone, we obtain the resonance values as
\[ j = -1, 0, 3, 4, -\gamma, \quad -\gamma \pm [ -2f(kl)]^{1/2}. \] (9)

From the careful analysis, we find that equation (9) admits sufficient number of positive resonances only for the condition
\[ -2f^2k = g \quad \text{and} \quad \gamma = -2 \] (10)

From the dominant term it is clear that \( \gamma = -2 \). The resonance \( r = -1 \) corresponds as usual to the arbitrariness of the singularity manifold and \( j = 0, 0 \) corresponds to the fact that either \( a_0 \) or \( b_0 \) and \( c_0 \) or \( d_0 \) are arbitrary in equation (8). Upon substituting the full Laurent series of equation (7) in equation (6) and on collecting the coefficients of different powers of \( \phi \) we find that equation (6) admits a sufficient number of arbitrary functions at \( j = 2, 3, 4, 4 \) and hence the system of equations (4) or (5) are expected to be integrable. It is very interesting to note that one can also obtain the same condition for the NLS–MB equations (4) from the expansion of the AKNS method.

With the substitution of the condition (10) in equation (4) we get the integrable NLS–MB equation as
\[ E_z = i[\frac{1}{2}E_{tt} + |E|^2E] + 2\rho, \]
\[ \rho_t = 2i\omega_0 \rho + 2E\eta, \] (11)
\[ \eta_t = -(E\rho^* + E^*\rho) \]

Once the Painlevé property is established, then one can systematically construct the Lax pair, the Bäcklund transformations and the soliton solutions from the Painlevé analysis by truncating the Laurent series at the constant level term. The construction of the Lax pair of a nonlinear partial differential equation will confirm the complete integrability of the same.

Here for completeness, we briefly discuss the Lax pair and the associated one-soliton solution of equation (11). The linear eigenvalue problem of equation (11) is found to be
\[ \psi_t = U\psi, \]
\[ \psi_z = V\psi, \] (12)
where
\[ U = \begin{bmatrix} \lambda E \\ -E^* \end{bmatrix}, \]
\[ V = i \begin{bmatrix} 1 & 0 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & E \\ -E^* & 0 \end{bmatrix} \lambda + \frac{1}{2} \begin{bmatrix} |E|^2 & E_t \\ E_t^* & |E|^2 \end{bmatrix} + \frac{1}{\lambda - i\omega_0} \begin{bmatrix} \eta & -\rho^* \\ -\rho & -\eta \end{bmatrix}, \]
where \( \lambda \) is the eigenvalue parameter. Now, we will discuss how one can generate the soliton solutions of equation (11). For this we use Darboux–Bäcklund transformation method. The key step in the construction involves the introduction of the function as
\[ \Gamma = \psi_1/\psi_2 \] (13)
So that the spatial and temporal linear eigenvalue problem reduces to the Riccati equations

\[ \Gamma_1 = E - 2i\lambda \Gamma + E^* \Gamma^2 \]
\[ \Gamma_2 = B + 2A \Gamma + B^* \Gamma^2 \]

where

\[ A = i(-2\lambda^2 + |E|^2) \]
\[ B = 2\lambda E + iE_t \]

Similarly, defining \( \Gamma' = \psi_1/\psi_2' \) will satisfy the equations similar to equation (14). Next we choose \( \Gamma' \) and \( E' \) as

\[ \Gamma' = \frac{1}{\Gamma^*}, \quad E' = E + \frac{2\Gamma^2 \Gamma'^* - \Gamma_1}{1 - |\Gamma|^4} \]

So that \( \Gamma' \) and \( E' \) satisfy equation (14) for the complex eigenvalue parameter \( \lambda = \nu + ip \). Now, equation (15) reduces to the following simplified form

\[ E' + E = \frac{-4\rho \Gamma}{1 + |\Gamma|^2} \]

where the primed quantities correspond to \( N \)-soliton solutions and the unprimed quantities correspond to \((N-1)\) soliton solutions. To construct the soliton solution of equation (16), we start with the zero soliton solution \( E(0) = 0, \rho = 0 \) and \( \eta = \pm 1 \) (pure states). By substituting the above conditions in the spatial and temporal eigenvalue problems, the explicit form of \( E(1) \) and \( \Gamma(0) \) are obtained. This procedure can obviously be continued and it furnishes in a recursive manner all the higher order soliton solutions and the associated wavefunction can also be generated. For instance, one soliton solution of equation (16) is found to be

\[ E(z, t) = 2\rho \sec h(x) \exp (iy - i\theta), \]
\[ \rho(z, t; \omega_0) = \frac{2\rho \{ \rho \sinh (x) + i(v - \omega_0) \cosh (x) \}}{\rho^2 \sinh (x) + (v - \omega_0)^2 \cosh^2 (x) + \rho^2/4}, \]
\[ \eta(z, t; \omega_0) = \frac{\rho^2 \sinh^2 (x) + (v - \omega_0)^2 \cosh^2 (x) - \rho^2/4}{\rho^2 \sinh^2 (x) + (v - \omega_0)^2 \cosh^2 (x) + \rho^2/4}, \]

where \( x(z, t) \) and \( y(z, t) \) are given by

\[ x(z, t) = 2\rho t + \left\{-4\rho v + \frac{2\rho}{\rho^2 + (v - \omega_0)^2}\right\} z + x^{(0)}, \]
\[ y(z, t) = 2\rho t + \left\{2(\rho^2 - v^2) - \frac{2(v - \omega_0)}{\rho^2 + (v - \omega_0)^2}\right\} z + y^{(0)}, \]

\( x^{(0)} \) and \( y^{(0)} \) are independent of both \( z \) and \( t \). Here \( \theta \) is a real constant and \( \rho, v \) are the velocity parameters associated with the soliton pulse.

From the nature of the solutions, it has been concluded that a stable \( 2\pi/N = 1 \) NLS-MB soliton exists. Also, the multisoliton structure shows that the higher-order NLS-MB solitons always split into multiple \( 2\pi/N = 1 \) solitons. In [3], these properties are also confirmed through a computer run. The phase change of the new soliton is governed solely by the NLS component and the pulse delay is determined solely by the SIT component when the detuning from the resonance is zero.
The complete details about the Painlevé analysis, the construction of the Lax pair, the Bäcklund transformation, the generation of soliton solutions and the behaviour of solitons through the numerical results for \(2f^2k \neq g\) will be published elsewhere.

3. Conclusion

Thus in this paper, by applying the Painlevé analysis, we have shown that the NLS–MB system describing the wave equation of optical pulses in fibres with dispersion, Kerr nonlinearity and impurities admits soliton-type propagation only for a particular value of the parameters. The entire system described with stimulated Raman scattering, stimulated Brillouin scattering, self-steepening and higher order dispersion are still under investigation.

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