DARK SOLITONS IN OPTICAL FIBERS WITH HIGHER ORDER EFFECTS

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We consider the higher order nonlinear Schrödinger (HNLS) equation, which governs the nonlinear wave propagation in optical fibers with higher order effects. Lax pair associated with the integrable HNLS equation for the pulse propagation in normal dispersion regime of the fiber media is constructed with the help of Ablowitz–Kaup–Newell–Segur method. Using Hirota bilinear method, dark soliton solution is explicitly derived. Similar study is also carried out for simultaneous propagation of \( N \) nonlinear pulses in the normal dispersion regime of the fiber system with higher order effects.

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1. Introduction

Hasegawa and Tappert,\(^1,2\) theoretically predicted the possibility of the propagation of envelope solitons in optical fibers and it was experimentally demonstrated by Mollenauer \( et al.\)\(^3\) in 1980. There onwards, numerous interesting research results, both theoretically and experimentally have been reported in the field of optical solitons, as they are very much useful in high speed digital optical fiber communication. In the absence of optical losses, wave dynamics of nonlinear pulses propagation in monomode fiber is described by the famous nonlinear Schrödinger (NLS) equation.\(^4,5\) The possibility of bright (dark) soliton in optical fibers is due to exact counter balancing between the effects, anomalous (normal) group velocity dispersion and self phase modulation.\(^1–6\)

Later, it has been found that the propagation of high bit rate soliton pulses are greatly influenced by higher order effects also.\(^4,5\) Physically important higher order effects are the higher order dispersion, self steepening, stimulated inelastic scattering and delayed nonlinear response. With the effect of all these physical processes, wave dynamics of optical fiber system is governed by the higher order NLS (HNLS) equation of the form,\(^4,5\)

\[
iqz - \frac{k''}{2}q_{TT} + \beta|q|^2q - \frac{ik'''}{6}q_{TTT} + i\gamma(|q|^2)Tq + i\gamma_s(|q|^2)Tq = 0, \tag{1}\]

197
where $q$ represents the complex envelope amplitude, $T$ and $Z$ are the time and distance along the direction of propagation, $k''$ is the second derivative of the axial wave number $k$ with respect to the angular frequency $\omega_0$ and describes group velocity dispersion, $k''' = \partial^3 k / \partial \omega^3$ at $\omega_0$ describes higher order dispersion, $\beta = n_2 \omega_0 / c A_{\text{eff}}$ is the self phase modulation parameter, where $n_2$ is the Kerr coefficient, and $c$ is the speed of light, $A_{\text{eff}}$ is the effective core area of the fiber, $\gamma = 2 \beta / \omega_0$ describes Kerr dispersion (also called self steepening) and $\gamma_s$ represents the delayed nonlinear process. Imaginary part of $\gamma_s$ describes stimulated Raman scattering. We consider only the real part of $\gamma_s$.

2. Higher Order Nonlinear Schrödinger Equation

Panilevé analysis of the HNLS Eq. (1) has been carried in many works. Through Painlevé analysis, conditions need to be satisfied for the integrability of Eq. (1) with all the higher order terms have been reported as

$$k''\gamma = \beta k''' \quad \text{and} \quad \gamma = -2\gamma_s. \quad (2)$$

These conditions are valid for both normal and anomalous dispersion regimes of the fiber system. In the fiber system Eq. (1), if $k''$ and $\beta$ are of opposite sign then it governs the pulse propagation in anomalous dispersion regime where the bright soliton exists. Without loss of generality if we consider $k'' = -\beta$ for bright soliton propagation, then the integrability conditions (2) become $\gamma = -2\gamma_s = -k'''$, and with variable transformations,

$$q(z, t) = u(z, t) \exp \left\{ i \left[ \frac{k''}{k'''} t - \frac{k'''}{k'''^3} z \right] \right\}, \quad z = \frac{-k''}{6} Z, \quad t = T - \frac{k''^2}{2k'''} Z, \quad (3)$$

Eq. (1) reduces to the following system of complex modified KdV type equation,

$$u_z + u_{ttt} + 6|u|^2 u_t + 3u(|u|^2)_t = 0. \quad (4)$$

Sasa and Satsuma were the first to report the inverse scattering transform (IST) scheme for Eq. (4). Soliton solutions using Bäcklund transformation and the Hirota bilinear method is presented in Ref. 8. For pulse propagation in normal dispersion regime of the fiber system Eq. (1), $k''$ and $\beta$ should be of identical sign, which is the condition for dark solitons. For dark soliton propagation if we consider $k'' = \beta$, then the integrability conditions (2) become $\gamma = -2\gamma_s = k'''$, and with variable transformations (3), Eq. (1) reduces to different complex modified KdV equation,

$$u_z - u_{ttt} + 6|u|^2 u_t + 3u(|u|^2)_t = 0. \quad (5)$$

Palacios et al., derived the dark soliton solution for the HNLS equation using the coupled amplitude-phase formulation. Here, we construct the Lax pair of the
integrable Eq. (5) using the Ablowitz–Kaup–Newell–Segur (AKNS) method\textsuperscript{12} as
\[
\frac{\partial \Psi}{\partial t} = U_1 \Psi, \quad \frac{\partial \Psi}{\partial z} = V_1 \Psi,
\]
where
\[
\Psi = (\psi_1 \ \psi_2 \ \psi_3)^T,
\]
\[
U_1 = \begin{pmatrix}
\zeta & 0 & iu \\
0 & \zeta & iu^* \\
-iu^* & -iu & -\zeta
\end{pmatrix},
\]
\[
V_1 = 4\zeta^2 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} + 4i\zeta^2 \begin{pmatrix}
0 & 0 & u \\
0 & 0 & u^* \\
-u^* & -u & 0
\end{pmatrix} - 2\zeta \begin{pmatrix}
|u|^2 & u^2 & -iu_t \\
-u^* & |u|^2 & -iu_t^* \\
-iu_t^* & -iu_t & -2|u|^2
\end{pmatrix}
\]
\[
+ \begin{pmatrix}
u u_t^* - u_t u^* & 0 & -4i|u|^2 + iu_{tt} \\
0 & u_t u^* - uu_t^* & -4i|u|^2 + iu_{tt}^* \\
4i|u|^2 - iu_{tt} & 4i|u|^2 - iu_{tt}^* & 0
\end{pmatrix}.
\]
Here $\zeta$ is the spectral parameter. Compatibility condition $U_{1z} - V_{1t} + [U_1, V_1] = 0$, gives the system Eq. (5). This proves the complete integrability of Eq. (5) which in turn proves the complete integrability of the HNLS fiber system for the nonlinear pulse propagation in the normal dispersion regime with the conditions (2).

In order to construct dark soliton solution using Hirota method, we apply the following form of bilinear transformation to Eq. (5),
\[
u = \frac{g(z, t)}{f(z, t)},
\]
where $g(z, t)$ is a complex function and $f(z, t)$ is a real function with respect to $z$ and $t$. Using Eq. (10), Eq. (5) can be decoupled into
\[
(iD_z - D_t^3 + 3\lambda D_t)(g \cdot f) = 0, \quad (D_t^2 - \lambda)f \cdot f = -4|g|^2, \quad D_t(g \cdot g^*) = 0.
\]
Here $D_z$ and $D_t$ are Hirota bilinear operators\textsuperscript{13} and $\lambda$ is a constant to be determined. Next for obtaining the dark soliton solution, we assume
\[
g = \tau(1 + \epsilon g_1), \quad f = 1 + \epsilon f_1,
\]
where $g_1$ and $f_1$ are complex and real functions of $z$ and $t$ and $\tau$ is a complex constant. Substituting (12) into (11) and collecting the coefficients of different powers of $\epsilon$, we derive the solutions
\[
g_1 = -f_1 = -\exp[m(t - \lambda z) + \xi^{(0)}],
\]
where $\lambda = m^2/2 = 4|\tau|^2$. Here $m$ and $\xi^{(0)}$ are real constants. Now using the solutions of $g_1, f_1$ and $\lambda$ in (12) and then in (10), the dark soliton solution can be derived as

$$u = \tau \exp(\pm i\pi) \tanh \left\{ \frac{1}{2} \left[ m \left( t - \frac{m^2}{2} z \right) + \xi^{(0)} \right] \right\}. \quad (14)$$

We find that the dark soliton solution of HNLS equation derived in Ref. 11 is same as the one derived here using the Hirota bilinear method. We also checked that the conditions reported in that work for the propagation of dark solitons in HNLS fiber system is the same integrability condition considered in this work for constructing the Lax pair and the dark soliton solution.

3. $N$-Coupled Higher Order Nonlinear Schrödinger Equation

Next we consider the simultaneous propagation of $N$ nonlinear waves in the fiber system with higher order effects. For this dynamics, the HNLS Eq. (1) can be written in the $N$ coupled form as

$$iq_j z - \frac{k''}{2} q_j T + \beta \sum_{n=1}^{N} |q_n|^2 q_j - \frac{i k''}{6} q_j T T + i \gamma \left( \sum_{n=1}^{N} |q_n|^2 q_j \right)_T \quad (15)$$

$$+ i \gamma_s \left( \sum_{n=1}^{N} |q_n|^2 \right)_T q_j = 0, \quad j = 1, 2, \cdots, N.$$

Two coupled form of Eq. (15) has been considered in Ref. 14. In that work they have derived the exact form of bright and dark soliton solutions using the Hirota bilinear method. Very recently, Painlevé analysis of the two coupled HNLS equation has been reported in Ref. 15. We have considered the two and three coupled integrable version of Eq. (15) and constructed the Lax pair and derived the bright soliton solutions using the Bäcklund transformation. \cite{16} The IST scheme for the $N$ coupled HNLS equations is reported in Ref. 17. Recently we have derived the bright soliton solutions for the integrable form of Eq. (15) using Bäcklund transformation. \cite{18} Here we construct the linear eigenvalue problem for the dark soliton case and derive the soliton solutions using the Hirota bilinear method. For $N$ fields propagation in the normal dispersion regime we consider the integrability conditions $k'' = \beta$ and $\gamma = -2\gamma_s = k'''$ and using the transformations

$$q_j (z, t) = u_j (z, t) \exp \left\{ i \left[ \frac{k''}{k'''} t - \frac{k'''}{k'''} z \right] \right\}, \quad (16)$$

$$z = -\frac{k''}{6} Z, \quad t = T - \frac{k''}{2k'''} Z,$$

Eq. (15) reduces to $N$ coupled complex modified KdV equations,

$$u_{jz} - u_{jttt} + 6 \sum_{n=1}^{N} |u_n|^2 u_{jt} + 3u_j \left( \sum_{n=1}^{N} |u_n|^2 \right)_t = 0. \quad (17)$$
Lax pair for $N$ coupled complex modified KdV equations (17) is derived using AKNS method as,

$$\frac{\partial \Psi}{\partial t} = U_2 \Psi, \quad \frac{\partial \Psi}{\partial z} = V_2 \Psi,$$

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_{2N+1} \end{pmatrix}^T,$$

where

$$U_2 = \begin{pmatrix} \zeta & 0 & \cdots & 0 & 0 & 0 & 0 & iu_N \\ 0 & \zeta & \cdots & 0 & 0 & 0 & 0 & iu_1^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \zeta & 0 & 0 & 0 & iu_2 \\ 0 & 0 & \cdots & 0 & \zeta & 0 & 0 & iu_2^* \\ 0 & 0 & \cdots & 0 & 0 & \zeta & 0 & iu_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \zeta & iu_1^* \\ -iu_N^* & -iu_N & \cdots & -iu_2^* & -iu_2 - iu_1 - \zeta \end{pmatrix},$$

$$V_2 = 4\zeta^3 \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{pmatrix} + 4i\zeta^2 \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1^* \\ -u_N^* & -u_N & \cdots & -u_2^* & -u_2 - u_1 - u_1 \end{pmatrix}.$$
where $B = \sum_{n=1}^{N} |u_n|^2$. Equation (17) can be obtained from the compatibility condition $U_{2z} - V_{2t} + [U_2, V_2] = 0$. This proves the complete integrability of Eq. (17) which in turn proves the complete integrability of the HNLS fiber system for the simultaneous propagation of $N$ nonlinear fields in the normal dispersion regime with the conditions (2).

In order to construct dark soliton solutions using Hirota method, we apply the following form of bilinear transformation to Eq. (17),

$$u_j = \frac{g_j(z, t)}{f(z, t)},$$  \hspace{1cm} (21)

where $g_j(z, t)$ are complex functions with respect to $z$ and $t$. Using Eq. (21), Eq. (17) can be decoupled into

\begin{align*}
(iD_z - D_t^3 + 3\lambda D_t)(g_j \cdot f) &= 0, \hspace{1cm} (22a) \\
(D_t^2 - \lambda) f \cdot f &= -4 \sum_{n=1}^{N} |g_n|^2, \hspace{1cm} (22b) \\
D_t(g_j \cdot g_{j+1}) &= 0, \hspace{1cm} (j = 1, \ldots, N - 1) \hspace{1cm} (22c) \\
D_t(g_N \cdot g_1) &= D_t(g_j \cdot g_j^*) = 0. \hspace{1cm} (22d)
\end{align*}

To obtain the dark soliton solutions, we assume

$$g_j = \tau_j (1 + \epsilon g_{1j}), \hspace{1cm} f = 1 + \epsilon f_1,$$  \hspace{1cm} (23)
where $\tau_j$ are complex constants and $g_{1j}$ are complex functions of $z$ and $t$. Substituting (23) into (22) and collecting the coefficients of different powers of $\epsilon$, we derive the solutions

$$g_j = -f_1 = -\exp[m(t - \lambda z) + \xi^{(0)}],$$

(24)

where $\lambda = m^2/2 = 4 \sum_{n=1}^{N} |\tau_n|^2$. Now using the solutions of $g_{1j}, f_1$ and $\lambda$ in (23) and then in (21), the dark soliton solutions can be derived as

$$u_j = \tau_j \exp(\pm i\pi) \tanh \left\{ \frac{1}{2} \left[ m \left( t - \frac{m^2}{2} z \right) + \xi^{(0)} \right] \right\}.$$  

(25)

In Ref. 14, similar dark soliton solutions for two coupled HNLS equations have been derived. But it has been found that for particular choice of parametric conditions there is a possibility of both dark and bright solitons propagating in both normal and anomalous dispersion regime of the optical fiber system. In that work for deriving the soliton solutions from the Hirota bilinear method they used only one condition $k''\gamma = \beta k''$. Then with the freedom available with the parameter $\gamma_s$, they had the possibility to discuss that both type of solitons can propagate in both type of dispersion regimes. We agree that their argument is correct from the availability of soliton solutions from the Hirota bilinear method with only one condition $k''\gamma = \beta k''$. But from the complete integrability conditions (2) both from Painlevé analysis$^{9,15}$ and from the above construction of the Lax pairs (for both single field and $N$ fields) we find that another condition $\gamma = -2\gamma_s$ must also be satisfied. This condition makes it clear that only dark soliton will propagate in the normal dispersion regime and only bright soliton will propagate in the anomalous dispersion regime even in the presence of higher order effects.

4. Conclusion

To conclude, we have considered the HNLS equation which describes the wave propagation of nonlinear optical field in fiber medium with important higher order effects. Using the integrability conditions for the pulse propagation in the normal dispersion regime and suitable variable transformations, HNLS equation has been transformed to a complex modified KdV type equation. Linear eigenvalue problem associated with the complex modified KdV equation has been constructed using AKNS method and exact form of dark soliton solution has also been derived using Hirota bilinear method. For the simultaneous propagation of $N$ nonlinear optical pulses in the normal dispersion regime of fiber transmission lines with higher order effects also we have constructed the Lax pair and derived the exact form of the dark soliton solutions. Finally we have shown that only dark (bright) solitons will propagate in the normal (anomalous) dispersion regime of the fiber system with higher order effects.
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